Episode 12

Free Vibration of Un-Damped Systems
Part 2

ENGN0040: Dynamics and Vibrations
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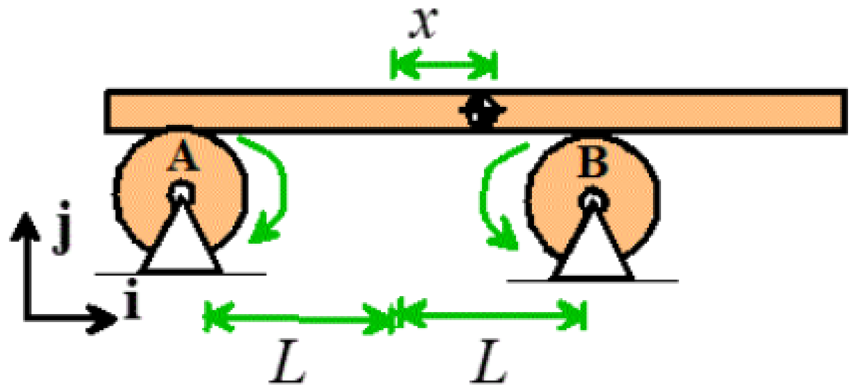
Topics for todays class

Calculating Natural Frequencies of Un-damped Systems

1. Using Newton's laws
2. Combining springs
3. Using energy methods to derive an EOM
4. Finding natural frequencies of nonlinear systems (pendulum, etc)

5.4.5 Example: The figure shows a 'friction oscillator.' The two wheels A and B spin rapidly so the contact between them and the bar slips at all times. The contacts have friction coefficient μ
Find a formula for the natural frequency.

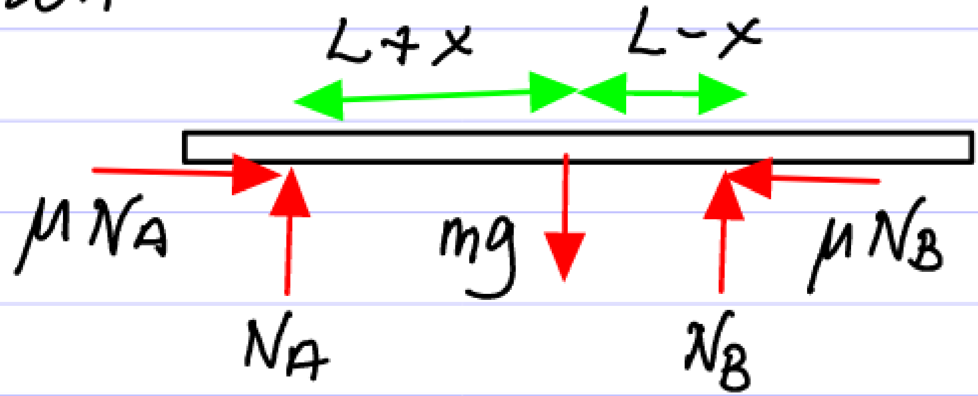


Approach (1) Find EOM

(2) Arrange in standard form

(3) Read off ω_n

FBD



$$\underline{F} = m\underline{a} \Rightarrow (\mu N_A - \mu N_B)\,\underline{i} + (N_A + N_B - mg)\,\underline{j} = m\frac{d^2x}{dt^2}\,\underline{i} \qquad (1)$$

$$\Sigma \underline{M}_{com} = \underline{0} \Rightarrow N_B(L-x) - N_A(L+x) = 0 \qquad (2)$$

Eliminate N_A, N_B:

(1) eq (2) $\Rightarrow$ $N_A - N_B = -(N_A + N_B)\, x / L$

(2) $\underline{j}$ comp of (1) $\Rightarrow$ $N_A + N_B = mg$

(3) $\underline{i}$ comp of (1) $\Rightarrow$ $\mu (N_A - N_B) = m \dfrac{d^2x}{dt^2}$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -\mu mg \frac{x}{L}$$

Hence $\underbrace{\dfrac{L}{\mu g}}_{\frac{1}{\omega_n^2}} \dfrac{d^2x}{dt^2} + x = 0$ "Standard form"

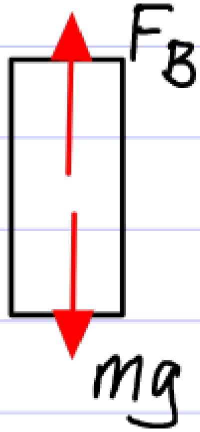
(from table)

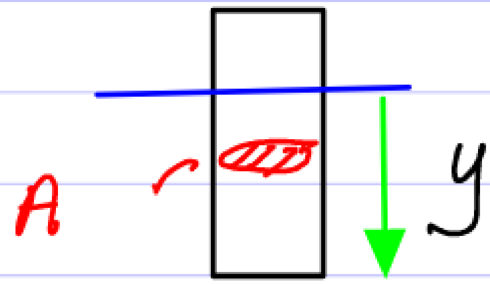
$$\Rightarrow \quad \boxed{\omega_n = \sqrt{\frac{\mu g}{L}}}$$

5.4.6 Example: A cylindrical buoy with diameter *D* and mass *m* floats in water with mass density ρ. Find a formula for the natural frequency.

Approach: (1) EOM; (2) Read off ω_n

FBD





$$F=ma \Rightarrow mg - F_B = m\frac{d^2y}{dt^2}$$

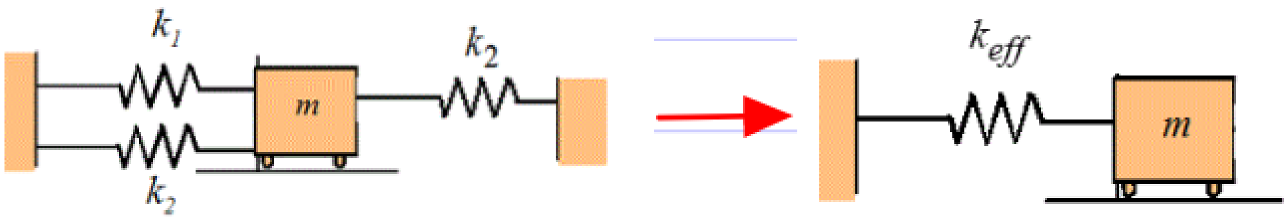
Archimedes $F_B = \text{Vol} \times \rho g = A y \rho g$

$$\Rightarrow m\frac{d^2y}{dt^2} = mg - \rho g A y \Rightarrow \underbrace{\frac{m}{\rho g A}}_{\frac{1}{\omega_n^2}}\frac{d^2y}{dt^2} + y = \frac{m}{\rho A}$$

$$A = \frac{\pi}{4}D^2 \Rightarrow \boxed{\omega_n = \sqrt{\frac{4m}{\pi D^2 \rho g}}}$$

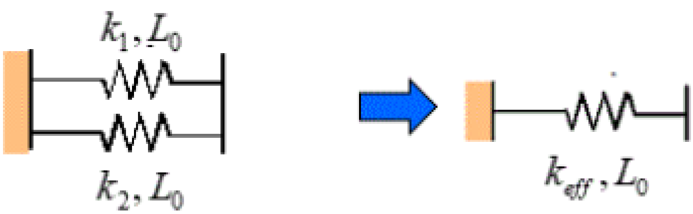
5.4.7 Shortcut for finding natural frequencies: Combining Springs

General idea: Replace a complex set of springs with a single spring & use formula for ω_n

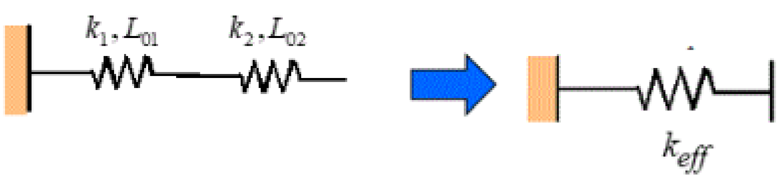


$$\omega_n = \sqrt{\frac{k_{eff}}{m}}$$

Formulas for effective stiffness



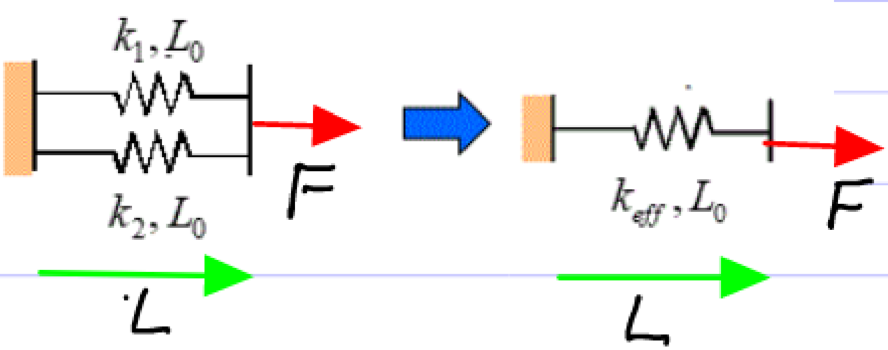
$$\boxed{\text{Parallel}; \quad k_{eff} = k_1 + k_2}$$



$$\boxed{\text{Series}: \quad \frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Derivation

Parallel springs (forces add)

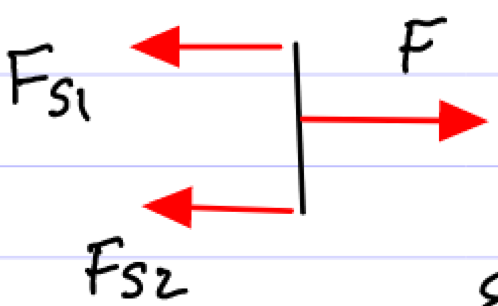


Spring Formula

$$F_{s1} = k_1 (L - L_0)$$

$$F_{s2} = k_2 (L - L_0)$$

Statics



$$F = F_{s1} + F_{s2} = k_1 (L - L_0) + k_2 (L - L_0)$$

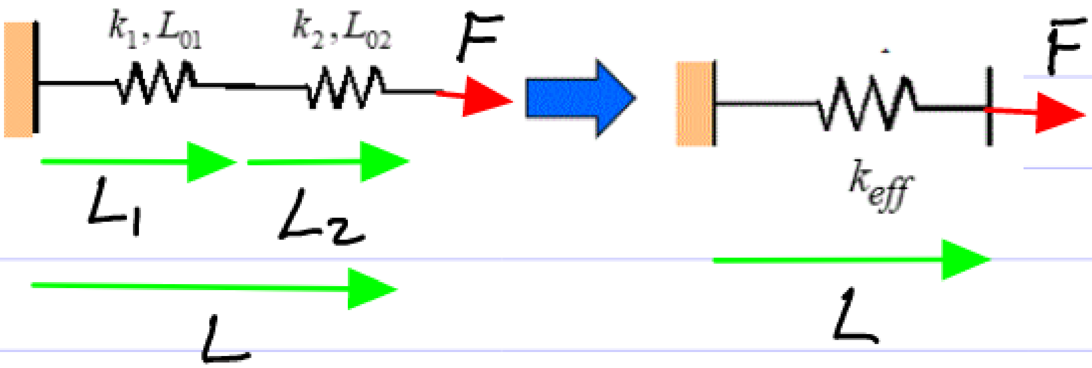
$$= (k_1 + k_2)(L - L_0)$$

Equal

Effective Spring $F = k_{eff} (L - L_0)$

Hence $k_{eff} = k_1 + k_2$

Series (lengths add)



Spring formulas

$$L_1 = L_{01} + F/k_1$$

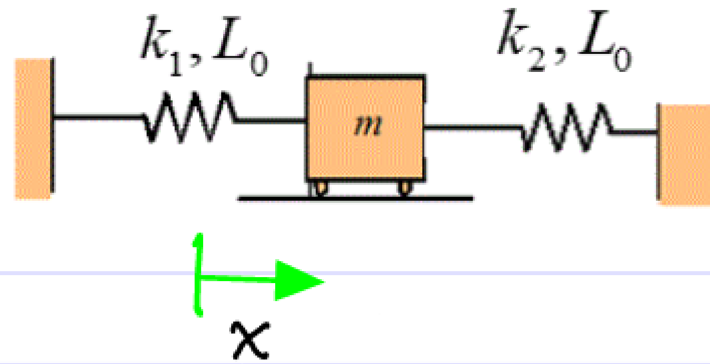
$$L_2 = L_{02} + F/k_2$$

Geometry $\quad L = L_1 + L_2 = L_{01} + L_{02} + F\left\{\frac{1}{k_1} + \frac{1}{k_2}\right\}$

Effective spring $\quad L = L_0 + F \cdot \frac{1}{k_{eff}}$

Hence $\quad \frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}$

Are these in series or parallel?

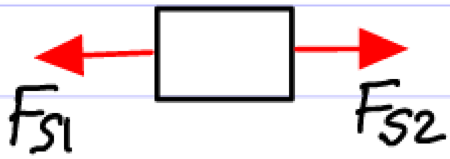


Find total force on mass

(displacement)

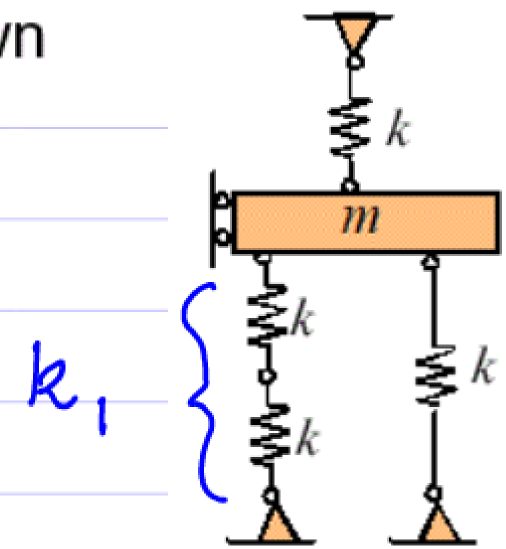
$F_{s1} = k_1 (L_0 + x - L_0)$

$F_{s2} = k_2 (L_0 - x - L_0)$



$\Rightarrow F = F_{s1} - F_{s2} = (k_1 + k_2)\, x \Rightarrow \boxed{\text{parallel!}}$

Example: Find the natural frequency of the system shown



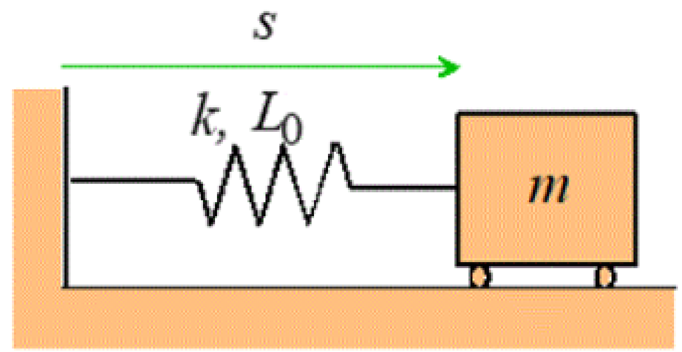
$\frac{1}{k_1} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_1 = k/2$

Other springs are in parallel

$\Rightarrow k_{eff} = 2k + \frac{k}{2} = \frac{5k}{2} \Rightarrow \boxed{\omega_n = \sqrt{\frac{5k}{2m}}}$

5.4.8 Using energy to derive an EOM

Re-visit spring-mass system



Conservative $\Rightarrow$ $T+U$ = constant

$$\frac{d}{dt}\left(T+U\right) = 0 \quad \Leftarrow \text{ Gives EOM!}$$

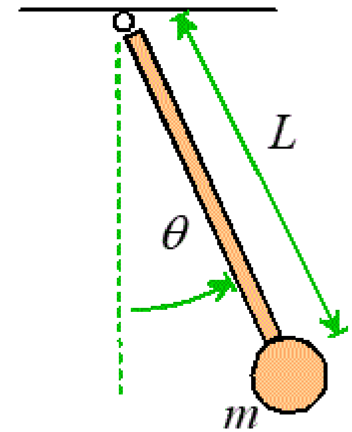
$$T = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 \qquad U = \frac{1}{2} k \left(s - L_0\right)^2$$

Use chain rule

$$\frac{d}{dt}(T+U) = m \cancel{\frac{ds}{dt}} \frac{d^2 s}{dt^2} + k(s-L_0)\cancel{\frac{ds}{dt}} = 0$$

$$\boxed{m\frac{d^2 s}{dt^2} + ks = kL_0}$$

5.4.9 Finding natural frequencies for nonlinear systems



Example: Find ω_n for pendulum

(1) Find EOM (use energy)

$$T = \frac{1}{2} m V^2 \qquad V = L\frac{d\theta}{dt} \qquad U = -mgL\cos\theta$$

$$\frac{d}{dt}\left\{T+U\right\} = \frac{d}{dt}\left\{\frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 - mgL\cos\theta\right\} = 0$$

$$\Rightarrow \quad mL^2 \cancel{\frac{d\theta}{dt}}\frac{d^2\theta}{dt^2} + mgL\sin\theta\cancel{\frac{d\theta}{dt}} = 0$$

$$\boxed{\frac{L}{g}\frac{d^2\theta}{dt^2} + \underbrace{\sin\theta}_{\text{Nonlinear}} = 0 \qquad \text{"Nonlinear" EOM}}$$

Pendulum is example of EOM of form

$$A\frac{d^2x}{dt^2} + f(x) = 0$$

Procedure :

(1) Assume small vibrations $x << 1$

(2) Expand $f(x)$ as Taylor series

$$f(x) = \underbrace{f(0)}_{\substack{\text{Always } = 0 \\ \text{for vibration EOM}}} + \underbrace{f'(0)}_{f' \equiv \frac{df}{dx}}\, x + \underbrace{\tfrac{1}{2} f''(0)\, x^2 + \cdots}_{\text{Small (neglect)}}$$

(3) Substitute in EOM

$$A\frac{d^2x}{dt^2} + f'(0)\, x = 0$$

$$\Rightarrow \underbrace{\frac{A}{f'(0)}}_{\frac{1}{\omega_n^2}} \frac{d^2x}{dt^2} + x = 0$$

$$\boxed{\omega_n = \sqrt{\frac{f'(0)}{A}}}$$

Apply to pendulum $\frac{L}{g}\frac{d^2\theta}{dt^2} + \sin\theta = 0$

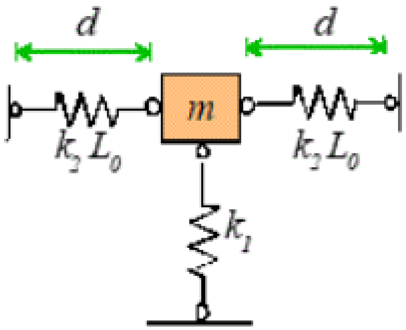
$$\sin\theta = \underbrace{\cancel{\sin}(0)}_{=0} + \underbrace{\cancel{\cos}(0)}_{=1}\,\theta + \underbrace{\cdots}_{\text{Neglect}}$$

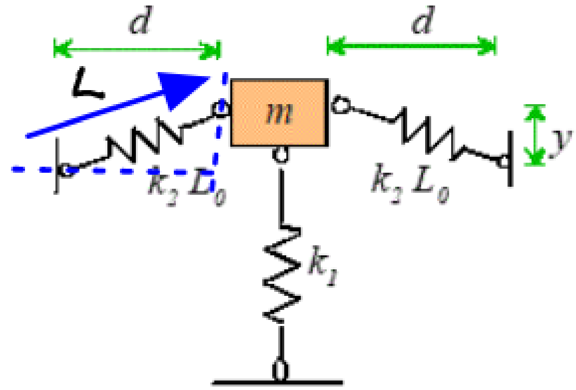
$$\underbrace{\frac{L}{g}}_{\frac{1}{\omega_n^2}}\frac{d^2\theta}{dt^2} + \theta = 0$$

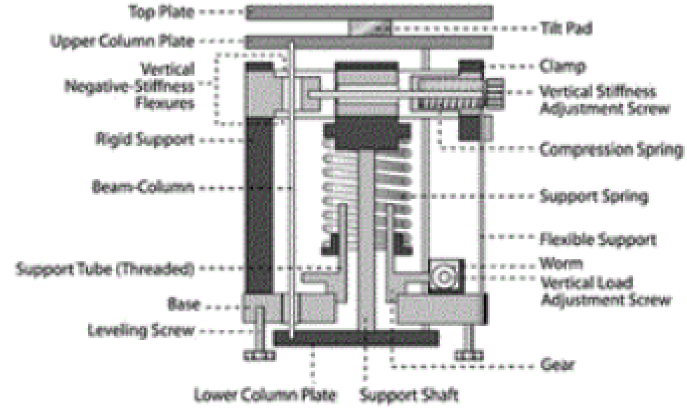
$$\boxed{\omega_n = \sqrt{\frac{g}{L}}}$$

Approximation accurate (< 5% error) for swing angles < 5° or so

5.4.10 Example: Find the natural frequency of the 'minus k' vibration isolation system (neglect gravity)







[Figure 3]

$$L = \sqrt{d^2 + y^2}$$

Approach

(1) Get EOM

(2) "Linearize" for small y

(3) Read off ω_n

$$T = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 \qquad U = \frac{1}{2} k_1 y^2 + 2 \times \frac{1}{2} k_2 (L - L_0)^2$$

$$\frac{d}{dt}(T+U) = m \frac{dy}{dt}\frac{d^2y}{dt^2} + k_1 y \frac{dy}{dt} + 2k_2 (L - L_0)\frac{dL}{dy}\frac{dy}{dt} = 0$$

Note $\dfrac{dL}{dy} = \dfrac{1}{2}\dfrac{1}{\sqrt{d^2+y^2}}\cdot 2y = \dfrac{y}{L}$

Hence EOM is

$$m\frac{d^2y}{dt^2} + \underbrace{k_1 y + 2k_2(L-L_0)\frac{y}{L}}_{f(y)} = 0$$

Linearize $f(y) = k_1 y + 2k_2\left(1-\dfrac{L_0}{L}\right)y$

$f(0) = 0$

$f'(y) = k_1 + 2k_2\left(1-\dfrac{L_0}{L}\right) + 2k_2\dfrac{L_0}{L^2}\dfrac{dL}{dy}\,y$

Note $L=d$ when $y=0$

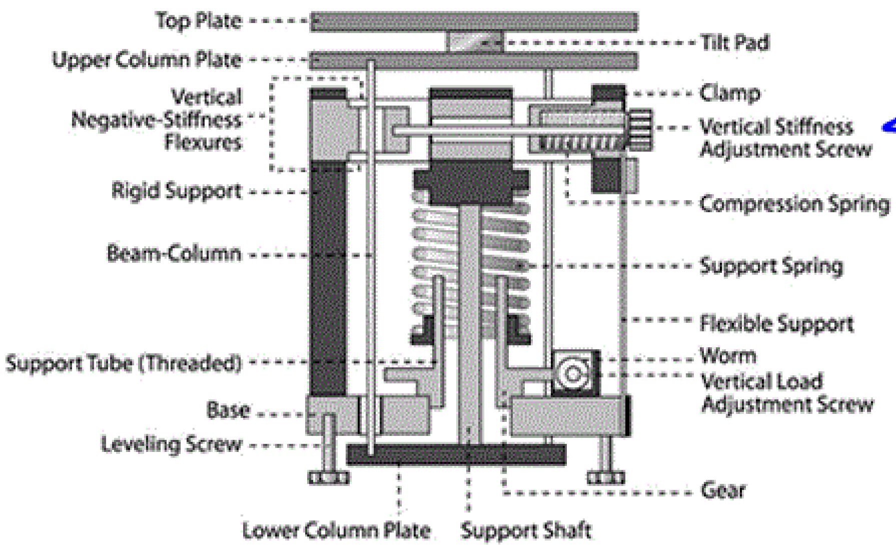
$$\Rightarrow \quad m\frac{d^2y}{dt^2} + \left\{k_1 + 2k_2\left(1-\frac{L_0}{d}\right)\right\}y = 0$$

Hence

$$\frac{1}{\omega_n^2}\quad \frac{m}{k_1 + 2k_2(1 - L_0/d)}\frac{d^2y}{dt^2} + y = 0$$

$$\boxed{\omega_n = \sqrt{\frac{k_1 + 2k_2\,(1 - L_0/d)}{m}}}$$

For good isolation we want ω_n close to zero



[Figure 3]

The "minus-k" design adjusts d to make $\omega_n \approx 0.5$ Hz

Used to adjust d

What happens if $k_1 + 2k_2\left(1 - \frac{L_0}{d}\right) < 0$

ω_n is complex ?

Re-write EOM (multiply through by -1)

$$\underbrace{\frac{m}{2k_2\left(\frac{L_0}{d}-1\right)-k_1}}_{>0} \frac{d^2y}{dt^2} - y = 0$$

$\frac{1}{\alpha^2}$

This is a "Case II" EOM: $\frac{1}{\alpha^2}\frac{d^2x}{dt^2} - x = -C$

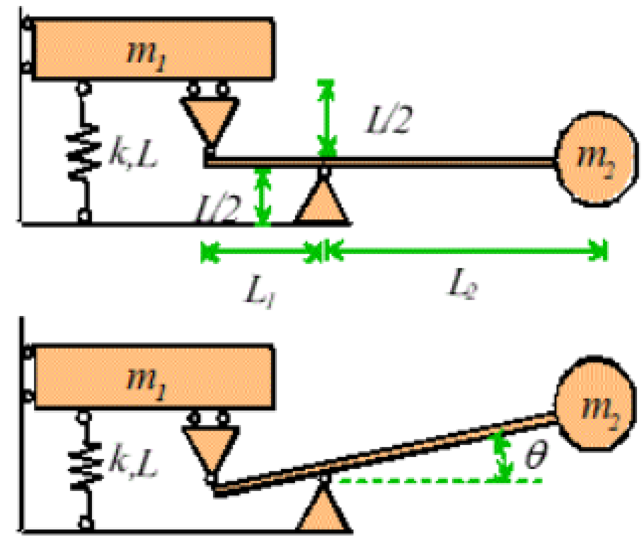
Solution has form

$$x = A\exp(\alpha t) + B\exp(-\alpha t)$$

No vibrations

Solution grows exponentially $\Rightarrow$ $\boxed{\text{unstable}}$

5.4.11 Example: Find the natural frequency of the anti-resonant vibration isolator (neglect gravity)



Find EOM using energy

$$T = \frac{1}{2} m_1 \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} m_2 \left(L_2 \frac{d\theta}{dt}\right)^2 \qquad y$$

$$U = \frac{1}{2} k (y - L)^2$$

Geometry: $y = \frac{L}{2} - L_1 \sin\theta + \frac{L}{2} = L - L_1 \sin\theta$

$$\Rightarrow dy/dt = -L_1 \cos\theta \, d\theta/dt$$

$$\Rightarrow T = \frac{1}{2} m_1 L_1^2 \cos^2\theta \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} m_2 L_2^2 \left(\frac{d\theta}{dt}\right)^2$$

$$U = \frac{1}{2} k L_1^2 \sin^2\theta$$

$$\frac{dT}{dt} = -m_1 L_1^2 \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta}\left(\frac{d\theta}{dt}\right)^3 + m_1 L_1^2 \cos^2\theta \frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$$
$$+ m_2 L_2^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$$

$$\frac{dU}{dt} = k L_1^2 \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta}\frac{d\theta}{dt}$$

$$\frac{d}{dt}(T+U) = 0 \Rightarrow$$
$$(m_1 L_1^2 \cos^2\theta + m_2 L_2^2)\frac{d^2\theta}{dt^2} - \frac{m_1 L_1^2}{2}\sin 2\theta\left(\frac{d\theta}{dt}\right)^2$$
$$+ \frac{k L_1^2}{2}\sin 2\theta = 0$$

Assume $\theta \ll 1$ $\quad d\theta/dt \ll 1$

$\Rightarrow \cos\theta \approx 1 \quad \sin\theta \approx \theta \quad \sin 2\theta \approx 2\theta$

$\sin 2\theta (d\theta/dt)^2 \approx 0$ (order θ^3)

Linearized EOM:

$$\frac{1}{\omega_n^2} \quad \frac{m_1 L_1^2 + m_2 L_2^2}{k L_1^2} \frac{d^2\theta}{dt^2} + \theta = 0$$

$$\boxed{\omega_n = \sqrt{\frac{k L_1^2}{m_1 L_1^2 + m_2 L_2^2}}}$$